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QUASICONFORMAL AND AFFINE GROUPS

GAVEN J. MARTIN

Introduction

Suppose that G is a discrete abelian group of diffeomorphisms acting on the unit sphere $\mathbf{S}^n$ of $\mathbf{R}^{n+1}$. The main result of this paper is that if G has uniformly bounded distortion and an element of infinite order, then G is conjugate, by a self-homeomorphism of $\mathbf{S}^n$ with bounded distortion, to a conformal group Γ of $\mathbf{S}^n$ (that is, Γ is a subgroup of the Möbius group). Actually, the restriction to abelian groups will be weakened to a class of admissible groups which will be defined by a simple algebraic condition (see §4). For instance, groups with an infinite cyclic central subgroup will be admissible; such groups can of course contain free groups of any rank. We will give a simple geometric condition on a subgroup of the euclidean group to be admissible. In [11], we showed that such a conjugacy exists in the case G is cocompact and isomorphic to a crystallographic group. Combining this with the results herein gives a wide class of abstract subgroups of the euclidean group for which any discrete and faithful representation in the diffeomorphism group of $\mathbf{S}^n$ with bounded distortion is conjugate into the euclidean group by a homeomorphism with bounded distortion.

We will provide a number of references from the recent literature to show how our results fit in with those obtained earlier. For instance, we recall from [11] that there is a uniformly quasiconformal group acting smoothly on $\mathbf{R}^n$ and isomorphic to a free abelian group of rank $n-1$ which is not quasiconformally conjugate to a euclidean group. Evidently it cannot be made smooth at infinity.

In order to study discrete groups of bounded distortion one is, of course, naturally led to the notion of a discrete quasiconformal group. For the basic facts regarding quasiconformal mappings we refer to Väisälä's book, [18] and for the theory of discrete quasiconformal groups we refer to the articles by Gehring and Martin [4] and Tukia [17].

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Our methods and ideas rest heavily on Tukia's work [17] on the conjugacy of certain uniformly quasiconformal groups to conformal groups using Sullivan's idea of measurable invariant conformal structures for quasiconformal groups (see also [14] and [7]). We make the easy but important observation that if every element of a uniformly quasiconformal group has constant matrix dilatation, then the averaged dilatation, which produces the invariant conformal structure, will also be constant and so can be realized as the pullback of the standard conformal structure via a linear mapping. This implies the following rigidity theorem in the affine group: a uniformly quasiconformal subgroup of the affine group is conjugate, via a linear mapping, to a conformal subgroup of the affine group. In particular this implies the well-known result that a finite or compact subgroup of $\mathrm{GL}(n, \mathbf{R})$ is linearly conjugate to a subgroup of the orthogonal group.

As an interesting consequence we will see that the properly discontinuous, fixed point free, affine action on $\mathbf{R}^3$ which is isomorphic to a free group on two generators, constructed by Margulis as a counterexample to a conjecture of Milnor (that such groups are virtually polycyclic) cannot have bounded distortion (see [10]). Indeed, it follows from our results that Milnor's conjecture is true if one assumes bounded distortion (for this will imply the group is virtually abelian). This last fact partially motivated our investigation.

Next, we will be able to show that the parabolic and loxodromic elements of a uniformly quasiconformal group are quasiconformally conjugate to Möbius transformations under some restrictions, for instance if the mapping is differentiable at a fixed point. We will also get good bounds on the dilatation of the conjugating mapping depending only on the dilatation of the corresponding cyclic group.

It had been previously observed that in dimensions $n \neq 4, 5$ the loxodromic elements of a uniformly quasiconformal group are topologically conjugate to Möbius transformations. This depends on the solution of the annulus conjecture and on certain results about fibering manifolds over the circle (see [4] and the references therein). Freedman's recent affirmative solution of the annulus conjecture in dimension 4 and 5 probably implies the result in these dimensions as well. It then follows from Sullivan's proof of the Hauptvermutung for quasiconformal manifolds in dimension $n \neq 4$ that this topological conjugacy can be made quasiconformal. However one cannot obtain any control on the dilatation of the conjugating map using these rather sophisticated and deep results. For parabolic transformations nothing was known except in dimension two, where the result is obvious from Sullivan's and Tukia's work (see [14] and [17]). We regard our method of approach to this problem as significantly less complicated, obtaining stronger results though in a less general setting.

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1. Notation, definitions and an estimate

We denote by $\overline{\mathbf{R}}^n$ the one point compactification of euclidean n-space $\mathbf{R}^n$. All notions of convergence will be taken with respect to the spherical metric of $\overline{\mathbf{R}}^n$.

We will often times prefer to work in $\overline{\mathbf{R}}^n$ rather than the conformally equivalent $\mathbf{S}^n$. Henceforth, we will assume that n is an integer which is at least two. We point out that almost all of our results are true (with essentially the same proofs) when $n = 1$, however there are complications due to the fact that there is no good definition of a pointwise dilatation in dimension one.

Whenever we claim an equation holds almost everywhere on $\mathbf{R}^n$, we mean almost everywhere with respect to n-dimension Lebesgue measure. In the theory of quasiconformal mappings, one is usually concerned with only equations defined and valid on sets of full measure.

We define the space $S(n) = \mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$. Thus $S(n)$ is the space of real, symmetric, positive definite $n \times n$ matrices with determinant equal to one. The general linear group $\mathrm{GL}(n, \mathbf{R})$ acts transitively on the right of $S(n)$ via the rule

$$X[A] = |\det X|^{-2/n} X^t A X, \qquad X \in \mathrm{GL}(n, \mathbf{R}),\ A \in S(n).$$

The Riemannian metric

$$ds^2 = \tfrac{1}{2}\sqrt{n}\,\mathrm{tr}(Y^{-1}\,dY)^2$$

on $S(n)$ gives rise to a metric distance which we denote by $d(A, B)$ for $A, B \in S(n)$. This metric is invariant under the right action of $\mathrm{GL}(n, \mathbf{R})$ and makes $S(n)$ a globally symmetric Riemannian manifold, which is complete, simply connected, and of nonpositive sectional curvature (see [8]). (We include the factor $\frac{1}{2}\sqrt{n}$ so that when $n = 2$ we obtain the usual hyperbolic space of constant curvature equal to -1.) We denote by Id the usual $n \times n$ identity matrix; then

$$d(A) = d(\mathrm{Id}, A) = \tfrac{1}{2}\sqrt{n}((\ln\lambda_1)^2 + (\ln\lambda_2)^2 + \cdots + (\ln\lambda_n)^2)^{1/2},$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A (see [9]). Other distances can now be calculated from the transitivity of the $\mathrm{GL}(n, \mathbf{R})$ action.

The *matrix dilatation* of a homeomorphism f of $\overline{\mathbf{R}}^n$, which is differentiable with nonzero Jacobian matrix almost everywhere, is the measurable map $\mu_f \colon \overline{\mathbf{R}}^n \to S(n)$ defined by

$$\mu_f(x) = |\det f(x)|^{-2/n} f'(x)^t f'(x) = f'(x)[\mathrm{Id}], \tag{1.1}$$

where $f'(x)$ is the Jacobian matrix of f.

A *conformal structure* on $\overline{\mathbf{R}}^n$ is a pair $(\overline{\mathbf{R}}^n, \mu)$, where $\mu\colon \overline{\mathbf{R}}^n \to S(n)$ is a measurable mapping for which

$$\operatorname{ess\,sup}\{d(\mu(x))\colon x \in \overline{\mathbf{R}}^n\} = d(\mu) < \infty.$$

We define $D(\mu) = \exp(d(\mu))$ and call this the *dilatation* of μ. The *standard conformal structure* is $\beta(x) \equiv \mathrm{Id}$. One should think of a conformal structure as a measurable ellipse field on the tangent space of $\overline{\mathbf{R}}^n$, such that the eccentricity of the ellipses is uniformly bounded. A homeomorphism of $\overline{\mathbf{R}}^n$ viewed as a mapping between two conformal structures μ_1 and μ_2,

$$f\colon (\overline{\mathbf{R}}^n, \mu_1) \to (\overline{\mathbf{R}}^n, \mu_2),$$

is called $D(\mu_1, \mu_2)$-*quasiconformal* if

(1) $f \in W^1_{n,\mathrm{loc}}(\overline{\mathbf{R}}^n)$, i.e. f has locally L^n integrable first derivatives,

(2) $D(f, \mu_1, \mu_2) = \operatorname{ess\,sup}\{\exp(d(\mu_1(x), f'(x)[\mu_2(f(x))]))\colon x \in \overline{\mathbf{R}}^n\} < \infty.$

If $\mu_1 = \mu_2 = \beta$, the standard structure, we obtain the usual notion of quasiconformality and if no conformal structures are present we mean quasiconformal in this usual sense. In this case the quantity $\exp(d(f'(x)[\mathrm{Id}]))$ is often referred to as the Ahlfors-Earle dilatation of a quasiconformal mapping at a point x (see [1]). Following Tukia, we will call the essential supremum of the above quantity over $\mathbf{R}^n$ the D-dilatation of f and we will denote this quantity $D(f)$. Notice that by definition $D(f) = D(\mu_f)$.

We say that f is conformal if $D(f) = 1$, that is,

$$\mu_1(x) = f'(x)[\mu_2(f(x))] \quad \text{almost everywhere in } \mathbf{R}^n.$$

For the basic facts concerning quasiconformal structures, see [17, §D]. One should notice that a mapping which is quasiconformal in one structure is automatically quasiconformal in all other structures, however the dilatation will vary from structure to structure. Furthermore, a map of $\overline{\mathbf{R}}^n$ which is one-quasiconformal in the usual sense is conformal in the usual sense and so a Möbius transformation.

The notion of D-dilatation is especially useful in our situation. The usual notion (that is of our reference [18]) of K-quasiconformality of a homeomorphism f of $\mathbf{R}^n$ is

$$K(f) = \operatorname{ess\,sup}\{\log(\lambda_{\max}(x)), \log(1/\lambda_{\min}(x))\}^{n/2},$$

where f satisfies (1) and (2), and $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ are the largest and smallest eigenvalues of the matrix $f'(x)^t f'(x) = f'(x)[\mathrm{Id}]$. Thus we state here the following sharp relation between the D-dilatation and the K-dilatation (see [17, (D5)].):

$$K(f)/\sqrt{n-1} \le D(f) \le \begin{cases} K(f), & n \text{ even}, \\ K(f)/\sqrt{(n-1)/n}, & n \text{ odd}. \end{cases}$$

Notice the implication that $K(f) = D(f)$ when $n = 2$.

1.2. Definition. A group G of homeomorphisms of $\overline{\mathbf{R}}^n$ is called a *quasiconformal group* if there is a finite K such that each element g of G is K-quasiconformal. We denote by $K(G)$ the infimum of such K. If each g in G is K-quasiconformal, then from (3) above $\{D(g)\colon g \in G\}$ is bounded and we denote by $D(G)$ the supremum of this set. Clearly the relation (3) is valid between $K(G)$ and $D(G)$.

For a quasiconformal group G, a *G-invariant conformal structure* μ on $\mathbf{R}^n$ is a conformal structure $(\overline{\mathbf{R}}^n, \mu)$ such that each g in G is conformal as a map

$$g\colon (\overline{\mathbf{R}}^n, \mu) \to (\overline{\mathbf{R}}^n, \mu).$$

We observe that if f is quasiconformal, then

$$f\colon (\overline{\mathbf{R}^n}, \mu_f) \to (\overline{\mathbf{R}}^n, \beta) \quad \text{and} \quad f^{-1}\colon (\overline{\mathbf{R}}^n, \beta) \to (\overline{\mathbf{R}}^n, \mu_f)$$

are conformal. Hence if μ is a G-invariant conformal structure and f is a homeomorphism satisfying (1) and (2), and $\mu_f(x) = \mu(x)$ almost everywhere, then the group $f \circ G \circ f^{-1}$ is a conformal group.

This is the basis for the proof that in two dimensions every uniformly quasiconformal group is quasiconformally conjugate to a conformal group. One can always produce a G-invariant conformal structure μ for a quasiconformal group and in dimension two the measurable Riemann mapping theorem (or the existence theorem for quasiconformal mappings) ensures that one can always find an f with $\mu_f = \mu$.

The existence of a G-invariant quasiconformal structure for a quasiconformal group G is due to the $\mathrm{GL}(n, \mathbf{R})$ invariance of the metric distance d in $S(n)$ and the fact that $S(n)$ is nonpositively curved. In fact if $P(E)$ denotes the center of the smallest ball containing a bounded subset E of $S(n)$ (which is well defined and unique since $S(n)$ is nonpositively curved and simply connected), then

$$\mu(x) = P(\{\mu_g(x)\colon g \in G\})$$

will be the desired G-invariant conformal structure (see [17, §§D and E]). Unfortunately, in higher dimensions ($n \ge 3$) there is no measurable Riemann mapping theorem (and there cannot be). Thus we will have to use different techniques to produce a conjugating mapping other than those suggested above for dimension two. However, we will need to produce invariant conformal structures and in order to produce the good bounds on the dilatation of the conjugating mapping that we claimed in the introduction it is necessary to prove the following technical lemma. In our application M will be the space $S(n)$. Given a metric ρ on a manifold M we denote by $B_\rho(z,t)$ the closed ρ-ball centered at z and of radius t.

1.3 Lemma. *Let M be a simply connected, nonpositively curved Riemannian manifold with complete metric $\rho(x,y)$. Let $x \in M$ and suppose that E is a subset of M containing x and such that* $\sup\{\rho(x,z) : z \in E\} \le s$. *Let $P(E)$ denote the center of the smallest ball containing E in the metric ρ. Then*

$$\rho(x, P(E)) \le s/\sqrt{2},$$

and this estimate is sharp (irrespective of the curvature of M).

Proof. Let $B_\rho(y,r)$ be the unique smallest ball containing the set E and suppose that $B_\rho(x,t)$ also contains E. Suppose that $\rho(x,y) = t/\sqrt{2} + c$ where $c > 0$. Let w be the point on the geodesic segment from y to x such that $\rho(y,w) = c/2$. Next let $z \in E$ and α be the angle formed by the geodesic segments wx and wz. By the law of cosines we see that if $\alpha \ge \pi/2$, then $\rho(z,w) \le t/\sqrt{2}$ while if $\alpha < \pi/2$, then $\rho(z,w) < r - \delta$, for some $\delta > 0$. Since we might as well assume that E is compact, we find that there is a positive δ such that E lies in the ball $B_\rho(w, \max\{t/\sqrt{2}, r-\delta\})$. This contradiction establishes the first part of the lemma.

To see that the estimate of the lemma is in general best possible, it suffices to observe that the value $s/\sqrt{2}$ is actually attained in the euclidean case (for instance if x is the vertex of a right isosceles triangle) and that the exponential map $\mathrm{Exp}_x : T_xM \to M$ is infinitesimally an isometry at the origin.

In the case of constant negative curvature equal to -1 (for instance in $S(2)$) one may use the formulas of hyperbolic trigonometry to prove the stronger estimate that $r = \min\{s/\sqrt{2}, (s+\log 2)/2\}$ (see [15]).

The above lemma significantly improves the estimates of the dilatation of the conjugating mapping in the theorems of [17] where the invariant conformal structure is assumed to be approximately continuous (respectively continuous) at a radial limit point (limit point) of the quasiconformal group G. These hypotheses are satisfied if, for instance, G is a quasiconformal group of $\mathbf{S}^n$ which extends to a quasiconformal group of $\overline{\mathbf{B}}^{n+1}$ with $\mathbf{B}^{n+1}/G$ compact. In the addendum to that paper, Tukia shows that under these hypotheses there is a quasiconformal mapping whose matrix dilatation is almost everywhere equal to the invariant conformal structure constructed above. This fact, together with our estimate (applied to the bounded subset $\{\mu_g : g \in G\}$ of $S(n)$ which always contains the identity) implies that the dilatation of the constructed conjugating map f in Tukia's Theorems G and H2 satisfies

$$d(\mu_f) \leq (1/\sqrt{2}) \sup\{d(\mu_g) \colon g \in G\},$$

and hence Tukia's theorem holds with the following better bounds on the conjugating map f:

$$\log D(G) \leq 2 \log D(f) \leq (1/\sqrt{2}) \log D(G).$$

We have also established the following theorem.

1.4 Theorem. *Let G be a quasiconformal group acting on an open subdomain of $\mathbf{R}^n$. Then there is a G-invariant conformal structure μ such that*

$$\log D(G) \leq 2 \log D(\mu) \leq \sqrt{2} \log D(G).$$

The first inequality holds for all G-invariant conformal structures and is sharp. The second inequality does not necessarily hold for all G-invariant conformal structures but is probably sharp for the μ that we construct. The referee suggested the following outline indicating this: Recall from computational geometry that for all m there is a constant c_m such that if X lies in $\mathbf{R}^m$ and $\operatorname{diam}(X) < 1$, then X lies in $B^m(x, c_m)$ for some x. Notice $c_2 = \sqrt{3}$ and for all m, $\sqrt{2} < c_m < \sqrt{3}$ and $c_m \to \sqrt{2}$ as $m \to \infty$. For simplicity, suppose $S(n)$ is flat. Then for all m, there is an n and matrices Id, $A_1, A_2, \cdots, A_m$ of $S(n)$ forming a regular m-simplex. If G is the cyclic group of order m acting on the disjoint union of m copies of $\mathbf{R}^n$, such that at some point z the invariant conformal structure is given by the average of the above matrices, then calculations reveal that

$$\log D(G(z)) = d(\mathrm{Id}, A_1),$$
$$\log D(\mu(x)) = \sqrt{((m+1)^2 - m - 1)/(2(m+1)^2)}$$

where $m = (n-1)(n+2)/2 = \dim S(n)$. Now let $n \to \infty$ and observe that although $S(n)$ is not flat, it is locally almost flat; also the disjoint union of m copies of $\mathbf{R}^n$ embeds in $\mathbf{R}^n$. This suggests that the estimate $\log D(\mu) < c_m^{-1} D(G)$ is actually sharp.

Using (3) above, one can translate Theorem 1.4 into an estimate on $K(f)$ as compared to $K(G)$. This sharpness of the lemma indicates that such an estimate is best possible using these techniques; it would be interesting to know whether these estimates on the dilatation of the conjugating mapping are sharp. As suggested above, there is evidence for this.

Finally, note that we will apply the lemma to the space $S(n)$ which as a symmetric space has dimension $(n-1)(n+2)/2$ and rank $n-1$. That is, there are $n-1$ dimensional flat subspaces. Hence the estimate of the lemma will be best possible in the large for $S(n)$, $n \geq 3$, and so one cannot hope to improve the general estimate for groups with large dilatation as one can in dimension two where $S(2)$ has rank one.

2. Quasiconformal and affine groups

2.1 Definitions and notation. We denote by $A(n) = \mathrm{GL}(n, \mathbf{R}) \times \mathbf{R}^n$ the affine group of $\mathbf{R}^n$ (an element is a homeomorphism of the form $Ax + b$, where $A \in \mathrm{GL}(n, \mathbf{R})$ and $b \in \mathbf{R}^n$). Möb(n) will denote the group of all conformal (or Möbius) transformations of $\overline{\mathbf{R}}^n$. We can extend $A(n)$ to act on $\overline{\mathbf{R}}^n$ by agreeing that every element should fix ∞. The group of similarities of $\mathbf{R}^n$ is that subgroup of the Möbius group fixing ∞; it is naturally isomorphic to $\mathbf{R} \cdot \mathrm{O}(n) \times \mathbf{R}^n$, where $\mathrm{O}(n)$ is the orthogonal group (every element is a homeomorphism of the form $rOx + b$, $r \in \mathbf{R} - \{0\}$, $O \in \mathrm{O}(n)$ and $b \in \mathbf{R}^n$).

2.2 Theorem. *Suppose that G is a quasiconformal group acting on a subdomain of $\mathbf{R}^n$ and that for each $g \in G$, μ_g is constant. Then G is conjugate to a conformal group by an affine mapping B whose D-dilatation $D(B)$ satisfies*

$$\log(D(G)) \le 2\log(D(B)) \le \sqrt{2}\log(D(G)).$$

Proof. The assumption is that μ_g is independent of x for an element g of G. So therefore is the (average) invariant conformal structure μ defined as above in §1. Thus $\mu(x) \equiv A \in S(n)$. By the transitivity of the $\mathrm{SL}(n, \mathbf{R})$ action, there is a B such that $A = B[\mathrm{Id}]$. Then the group $H = B^{-1} \circ G \circ B$ is conformal in the usual sense.

In view of the estimates of 1.2(3), we find the estimate

$$K^{1/2} \le K(f) \le \sqrt{(n-1)}K^{1/\sqrt{2}}$$

is also valid.

2.3 Remarks. Actually, the assumption that every element of a group G of homeomorphisms of $\overline{\mathbf{R}}^n$ have constant matrix dilatation implies that either G is a subgroup of the affine group or else G is a conformal group. One can see this from a direct, but tedious, calculation or otherwise from the above theorem $G = A \circ H \circ A^{-1}$, where H is a conformal group and A is linear. If every element of H fixes infinity, then G is the affine conjugate of a similarity group and so an affine group. Otherwise, there is an h in H which does not fix infinity. Let $g = A \circ h \circ A^{-1}$ and notice that $g(\infty) \neq \infty$. Our assumption is that g has constant matrix dilatation. Now there is a conformal mapping σ such that $\sigma \circ g(\infty) = \infty$. Since $\mu_{\sigma g} = \mu_g$ is constant, we see $\sigma \circ g = C$, where C is some affine transformation. Every conformal mapping which does not fix infinity can be written in the form $\varphi \circ S$, where S is a similarity and

φ is the usual inversion in the unit sphere. Equating the two expressions thus obtained for g we find there is an affine mapping D such that $A \circ \varphi = \varphi \circ D$. This implies that D is linear, as A is, and that

$$Ax = (|Dx|/|x|)^2 Dx \quad \text{for } x \neq 0.$$

The left-hand side is smooth at $x = 0$, so therefore is the right-hand side. This is impossible unless $|Dx| = a|x|$ for some real positive constant a. Thus D and hence A are conformal. So too then is the group G.

Furthermore, it is only necessary to assume that every element of G have matrix dilatation which is almost everywhere, with respect to Lebesgue measure, the same constant matrix.

It is a rather unfortunate fact that a homeomorphism f of $\overline{\mathbf{R}}^n$ may have constant matrix dilatation while f^{-1} does not. Indeed, the above argument implies that if both f and f^{-1} have constant matrix dilatation, then either f fixes infinity and is affine or else f is conformal.

2.4 Corollary. *Let G be a subgroup of the affine group. If $d(G) = \sup\{d(\mu_g)\colon g \in G\}$ is finite, then G is conjugate to a group of similarities by a linear mapping B with*

$$d(G)/2 \leq d(B) \leq d(G)/\sqrt{2}.$$

2.5 Corollary. *Let G be a finite or compact subgroup of $A(n)$ or $\mathrm{GL}(n,\mathbf{R})$. Then there is a linear transformation A of $\mathrm{GL}(n,\mathbf{R})$ such that AGA^{-1} is contained in $\mathrm{O}(n)$ and*

$$d(G)/2 \leq d(A) \leq d(G)/\sqrt{2}.$$

Proof. The map $\xi\colon A(n) \to S(n)$, defined by $\xi(B) = B^t B$, is clearly continuous. So if G is finite or compact, then $\xi(G)$ is finite or compact in $S(n)$ and so in particular $d(G)$ is bounded.

As a consequence of the above corollaries, any uniformly quasiconformal subgroup of $A(n)$ which acts freely (without fixed points) on $\mathbf{R}^n$ is conjugate to a subgroup of euclidean isometries via a linear mapping. In particular, if a discrete quasiconformal subgroup G of $A(n)$ acts freely and cocompactly on $\mathbf{R}^n$ (that is, $\mathbf{R}^n/G$ is compact), then G is conjugate to a Bieberbach (or crystallographic) group via a linear mapping. Consequently any free uniformly quasiconformal subgroup G of $A(n)$ is virtually abelian (that is, contains an abelian subgroup of finite index).

Milnor conjectured (see [10]) that a fixed point free, discontinuous subgroup of $A(n)$ is virtually polycyclic (that is it contains a polycyclic subgroup of finite index). Milnor's conjecture is evidently true for uniformly quasiconformal subgroups of $A(n)$. Thus any example such as that of Margulis

[10] of a subgroup of the affine group which is fixed point free, properly discontinuous and isomorphic to a free group of rank two cannot be uniformly quasiconformal.

We note here that the above results are well known in the case that the group is a finite or compact subgroup of $A(n)$ (for instance, a continuous representation of a compact group in $\mathrm{GL}(n, \mathbf{R})$ is conjugate to an orthogonal representation, actually amenability suffices; see [6]). However we do not know whether such (or any) bounds on the norm of the conjugating linear mapping have ever been found.

As an application of these methods we reformulate some of the above results. For a matrix $A \in \mathrm{GL}(n, \mathbf{R})$ we set

$$\|A\| = \max\{|Ah| \colon |h| = 1\}.$$

We say A is similar to B if there is a C in $\mathrm{SL}(n, \mathbf{R})$ such that $A = C^{-1}BC$.

As corollaries to our earlier results we see

2.6 Corollary. *Let F be a subgroup of $\mathrm{GL}(n, \mathbf{R})$. Suppose there is a constant $c \geq 1$ such that for each A in the group F*

$$\|A\|^n \leq c \det A. \tag{2.7}$$

Then there is an element B of $\mathrm{SL}(n, \mathbf{R})$ with $\|B\| \leq \sqrt{(n-1)}c^{1/\sqrt{2}}$, such that the group $B^{-1}FB$ lies in the similarity group $\mathbf{R} \cdot \mathrm{O}(n)$.

A linear operator is said to be power bounded if there is a constant c such that for every positive m,

$$\|A^m\| \leq c. \tag{2.8}$$

2.9 Corollary. *The matrix A is power bounded if and only if A is similar to an orthogonal transformation.*

We observe that in Corollary 2.9 the hypotheses imply that $\lim \|A^m\|^{1/m} = 1$, that is the spectral radius of A is one. Since $\det A = 1$, all the eigenvalues of A have modulus one. This alone, however, does not suffice to assert the validity of the conclusion.

3. The infinite cyclic subgroups of a smooth quasiconformal group

In [4] we show that every element of a quasiconformal group is either elliptic, loxodromic or parabolic (see the definitions below). Due to the affirmative solution of the Smith conjecture, it follows that every periodic diffeomorphism of $\overline{\mathbf{R}}^n$, $n \leq 3$, is smoothly conjugate to an orthogonal transformation if it has fixed points (see [13]). For $n \geq 4$, the failure of the generalized Smith Conjecture provides examples of smooth, periodic diffeomorphisms which are not topologically conjugate to orthogonal transformations (the fixed point sets can be made to be codimension two knotted spheres; see [5]). Such finite groups of diffeomorphisms are examples of uniformly quasiconformal groups which are not topologically conjugate to any conformal groups. We will show here, however, that the elements of infinite order in a smooth discrete quasiconformal group are quasiconformally conjugate to Möbius transformations in all dimensions. We do not know if the assumption of smoothness is necessary (it is not for the loxodromic transformations when $n \neq 4$).

The argument providing the conjugacy is similar in both the parabolic and loxodromic cases and we will need both cases to attack the general question in §4.

We need some terminology in order to state the best results.

3.1 Definition. A self-homeomorphism f of $\mathbf{R}^n$ is said to be *affine at infinity* if f is differentiable almost everywhere and there is a matrix A of $S(n)$ such that

$$d(\mu_f(x), A) \to 0 \quad \text{as } x \to \infty, \tag{3.2}$$

where d is the metric of $S(n)$ as defined above in §1 (notice that d generates the usual topology of $S(n)$ as a space of matrices). We will say that f is *conformal at infinity* if A is the identity.

Affine mappings will naturally be affine at infinity. However, of course, affine maps need not be differentiable at infinity.

3.3 Lemma. *If f is continuously differentiable at infinity with nonzero Jacobian, then f is affine at infinity.*

Proof. As above we will continue to denote the usual inversion in the unit sphere by φ. The hypothesis that f is continuously differentiable at infinity with nonzero Jacobian means that $g = \varphi \circ f \circ \varphi$ is continuously differentiable at the origin with nonzero Jacobian. Thus there is an $A \in S(n)$,

$$A = (\varphi \circ f \circ \varphi)'(0)^t (\varphi \circ f \circ \varphi)'(0) = g'(0)[\mathrm{Id}]$$

such that

$$d(\mu_g(x), A) \to 0 \quad \text{as } x \to 0.$$

Now $\mu_g(x) = \mu_{f\varphi}(x) = \varphi'(x)[\mu_f(\varphi(x))]$ and since this right action (the right $\mathrm{GL}(n, \mathbf{R})$ action on $S(n)$) is isometric we see

$$d(\varphi'(x)[\mu_f(\varphi(x))], A) = d(\mu_f(\varphi(x)), (\varphi'(x))^{-1}[A])) \to 0 \quad \text{as } x \to 0.$$

Next an easy calculation reveals $\varphi'(x)$ is the conformal matrix $\mathrm{Id} - 2Q(x)$, where $Q(x) = (q_{ij}(x))$ and $q_{i,j} = |x|^{-4}(x_i x_j)$. Since $x = \varphi \circ \varphi(x)$, $\mathrm{Id} = \varphi'(\varphi(x))\varphi'(x)$ and so $\varphi'(x)^{-1} = \varphi'(\varphi(x)) \to \mathrm{Id}$ as $x \to 0$. Thus $(\varphi(x))^{-1}[A] \to A$ as $x \to 0$.

Finally, if we set $y = \varphi(x)$ we see $d(\mu_f(y), A) \to 0$ as $y \to \infty$, thus establishing the lemma.

3.4 Definition. A quasiconformal homeomorphism f of $\overline{\mathbf{R}}^n$ is called *parabolic* if f has a single fixed point and the group generated by f, $\langle f \rangle = \{f^m : m \in \mathbf{Z}\}$, is infinite, discrete and quasiconformal. In this case

$$f^{\pm m} \to x_0 \text{ locally uniformly in } \overline{\mathbf{R}}^n - \{x_0\} \text{ as } m \to \infty, \tag{3.5}$$

where x_0 is the fixed point of f (see [4]). We set $D(\langle f \rangle)$ as the D-dilatation of the quasiconformal group $\langle f \rangle$. If $x_0 = \infty$, we call f a parabolic quasiconformal homeomorphism of $\mathbf{R}^n$.

At present it is unknown whether in dimension three or more a parabolic quasiconformal homeomorphism of $\mathbf{R}^n$ is quasiconformally (or even topologically) conjugate to a euclidean isometry. This is true in dimension one and two. There are examples of homeomorphisms of $\mathbf{R}^n$, $n \geq 3$, satisfying (3.5) with $x_0 = \infty$, which are not topologically conjugate to a euclidean isometry (see [4]). These examples are not quasiconformal however and from our next theorem cannot even be topologically conjugate to a quasiconformal homeomorphism which is differentiable or affine at infinity.

We need to recall Tukia's generalization of the good approximation theorem (see [17, Corollary D].).

Theorem (*Good approximation theorem*). *Let $f_i : U \to \mathbf{R}^n$ be a sequence of K-quasiconformal embeddings. Suppose that $f_i \to f$ for some embedding $f : U \to \mathbf{R}^n$ and that $\mu_{f_i} \to \mu$ in measure for some measurable map $\mu : U \to S(n)$. Then f is K-quasiconformal and $\mu_f = \mu$ a.e. in U.*

3.6 Theorem. *Let f be a parabolic quasiconformal homeomorphism of $\mathbf{R}^n$ which is affine at infinity. Then f is conjugate to a parabolic Möbius transformation by a K-quasiconformal mapping for which*

$$K(\langle f \rangle)^{1/2} \leq K \leq \sqrt{(n-1)}K(\langle f \rangle)^{1+1/\sqrt{2}}. \tag{3.7}$$

Proof. Equation (3.5) implies that $f^{\pm m} \to \infty$ as $m \to \infty$. Choose similarity maps $\beta_m = a_m x + b_m$, where a_m is real and b_m lies in $\mathbf{R}^n$, such that

$$f^m \circ \beta_m(0) = 0 \quad \text{and} \quad |f^m \circ \beta_m(e_1)| = 1,$$

where $e_1 = (1, 0, \cdots, 0)$ is one of the usual basis vectors of $\mathbf{R}^n$. Now since the mapping f^m is $K(\langle f \rangle)$-quasiconformal for all m and since every β_m is conformal, the sequence $\{f^m \circ \beta_m\}$ is a normalized sequence of $K(\langle f \rangle)$-quasiconformal mappings, and so contains a uniformly convergent subsequence $\{f^{m(j)} \circ \beta_{m(j)}\}$ converging to a $K(\langle f \rangle)$-quasiconformal homeomorphism h of $\mathbf{R}^n$ as $j \to \infty$ (see for instance [18]).

We claim $g = h^{-1} \circ f \circ h$ generates a $K(\langle f \rangle)$-quasiconformal group of affine transformations. To see this note that for all integers k

$$\begin{aligned} g^k &= h^{-1} \circ f^k \circ h = \lim \beta_{m(j)}^{-1} \circ f^{-m(j)} \circ f^k \circ f^{m(j)} \circ \beta_{m(j)} \\ &= \lim \beta_{m(j)}^{-1} \circ f^k \circ \beta_{m(j)} \end{aligned}$$

as $j \to \infty$, by the uniform convergence of the sequence $f^{m(j)} \circ \beta_{m(j)}$. Since the β_m are conformal and since f^k is $K(\langle f \rangle)$-quasiconformal, we find from the right-hand side of the above equation that g generates a $K(\langle f \rangle)$-quasiconformal group.

We now need to show that g is in fact affine. To do this we will compute its matrix dilatation and show that it is constant (note $g(\infty) = \infty$).

Let $g_j = \beta_{m(j)}^{-1} \circ f \circ \beta_{m(j)}$ so that

$$\begin{aligned} \mu_{g_j}(x) &= \mu_{f \circ \beta_{m(j)}}(x) = \beta'_{m(j)}[\mu_f(\beta_{m(j)}(x))] \\ &= \mu_f(\beta_{m(j)}(x)). \end{aligned}$$

We now need to observe that

$$\beta_{m(j)}(0) = f^{-m(j)}(0) \to \infty \quad \text{and} \quad \beta_{m(j)}(e_1) \in f^{-m(j)}(\mathbf{S}^{n-1}) \to \infty$$

so that from the Carathéodory convergence theorem (see [8]) $\beta_{m(j)} \to \infty$ locally uniformly in $\mathbf{R}^n$. The assumption that f is affine at infinity now implies that there is an $A \in S(n)$ such that

$$\mu_{g_j}(x) = \mu_f(\beta_{m(j)}(x)) \to A \quad \text{as } j \to \infty.$$

We now are in the situation that $g_j \to g$ uniformly in $\mathbf{R}^n$ and the matrix dilatations of the g_j also converge uniformly to a matrix $A \in S(n)$. From the good approximation theorem we conclude that $\mu_g \equiv A$. That is, g has constant matrix dilatation. Hence g is affine and so too therefore is the group generated by g.

Finally, to conclude the proof, we find from Theorem 2.2 that the affine, uniformly quasiconformal group $\langle g \rangle$ is quasiconformally conjugate to a Möbius group by a linear mapping B whose dilatation is no more than $\sqrt{(n-1)}K(\langle g \rangle)^{1/\sqrt{2}}$. Thus,

$$B^{-1} \circ h^{-1} \circ f \circ h \circ B$$

is Möbius, so that $h \circ B$ is the desired $(\sqrt{(n-1)}K(\langle f \rangle)^{1+1/\sqrt{2}})$-quasiconformal mapping conjugating f to a parabolic Möbius transformation and thereby establishing the theorem.

We could simplify and improve the estimate on the dilatation of the conjugating map in the theorem if we knew that the limit of a sequence of quasiconformal mappings with D-dilatation bounded by M and converging uniformly to a homeomorphism was again quasiconformal with D-dilatation bounded by M. Tukia has informed us that at present this is unknown.

Observe that we only need that the matrix dilatations of g_j converge in measure to A, as the good approximation theorem will still apply. However it seems difficult to give an easy condition (other than the one we have) to guarantee this.

3.8 Corollary. *If f is a parabolic quasiconformal diffeomorphism of $\overline{\mathbf{R}}^n$, then f is conjugate to a euclidean isometry by a quasiconformal homeomorphism satisfying the dilatation estimate* (3.7).

As a consequence of Corollary 3.8 a parabolic quasiconformal diffeomorphism of $\mathbf{S}^n$ is quasiconformally conjugate to a Möbius transformation of $\mathbf{S}^n$.

3.9 Corollary. *If f is a parabolic quasiconformal homeomorphism of $\mathbf{R}^n$ which is conformal at infinity, then f is conjugate to a euclidean isometry by a K-quasiconformal mapping for which $K(\langle f\rangle)^{1/2} \leq K \leq K(\langle f\rangle)$.*

Proof. We observe that if f were conformal at infinity, then in the proof of Theorem 3.7 where we established that g was affine (the matrix dilatation of g was equal to A) we would have shown the matrix dilatation of g was the identity and so g would already have been a euclidean isometry.

3.10 Definition. A quasiconformal homeomorphism f of $\overline{\mathbf{R}}^n$ is called *loxodromic* if f has exactly two distinct fixed points and the group $\langle f\rangle$ is infinite, discrete and quasiconformal. In this case we may label the fixed points x_0 and y_0 so that

$$f^m \to x_0 \quad \text{locally uniformly in } \overline{\mathbf{R}}^n - \{y_0\},$$
$$f^{-m} \to y_0 \quad \text{locally uniformly in } \overline{\mathbf{R}}^n - \{x_0\},$$

as $m \to \infty$.

We call x_0 the *attractive* fixed point of f and y_0 the *repulsive* fixed point.

For the definition of approximate continuity (or continuity in measure) of a measurable map we refer to [2]. It is worthwhile recalling that a measurable map is approximately continuous almost everywhere.

It is known that for all $n \neq 4$, a loxodromic quasiconformal transformation of $\overline{\mathbf{R}}^n$ is quasiconformally conjugate to a loxodromic Möbius transformation. The conjugacy is known only to be topological when $n = 4$ (see [4]). These facts require remarkably deep results in topology and analysis as we mentioned in the introduction, namely the annulus theorem in all dimensions and the quasiconformal Hauptvermutung for $n \neq 4$. Moreover, this approach yields no bounds at all on the dilatation, so that for instance in a quasiconformal group containing loxodromics with distinct conjugacy classes, one can say very little about the structure of the elements. Thus we will now prove the following theorem in an analogous fashion to Theorem 3.8.

3.11 Theorem. *Let f be a loxodromic quasiconformal transformation of $\overline{\mathbf{R}}^n$ whose matrix dilatation is approximately continuous at a fixed point. Then f is conjugate to a loxodromic Möbius transformation by a K-quasiconformal mapping for which*

$$K(\langle f\rangle)^{1/2} \leq K \leq \sqrt{(n-1)}K(\langle f\rangle)^{1+1/\sqrt{2}}.$$

Proof. In this case we may assume by conjugating f by a conformal mapping and replacing f with f^{-1} if necessary, that 0 is the attractive fixed point of f and ∞ is the repulsive fixed point and that the matrix dilatation of f is approximately continuous at 0. Let $\beta_m(x) = a_m x$, where the a_m are real and positive and such that the sequence $\{f^m \circ \beta_m\}$ satisfies $|f^m \circ \beta_m(e_1)| = 1$. Since $f^m \circ \beta_m(0) = 0$, the sequence $\{f^m \circ \beta_m\}$ is a normalized sequence of $K(\langle f\rangle)$-quasiconformal mappings of $\mathbf{R}^n$ from which we may extract a uniformly convergent subsequence $\{f^{m(j)} \circ \beta_{m(j)}\}$ converging to a $K(\langle f\rangle)$-quasiconformal homeomorphism h of $\mathbf{R}^n$. Notice that $\beta_{m(j)} \to \infty$ locally uniformly in $\mathbf{R}^n - \{0\}$. Let $g = h \circ f \circ h^{-1}$ and define

$$g_j = \beta_{m(j)} \circ f^{m(j)} \circ f \circ f^{-m(j)} \circ \beta_{m(j)}^{-1} = \beta_{m(j)} \circ f \circ \beta_{m(j)}^{-1}.$$

By the uniform convergence of the sequence $\beta_{m(j)} \circ f^{m(j)}$ we see $g_j \to g$ and computing the matrix dilatations we find (since $a_m \to \infty$)

$$\mu_{g_j}(x) = \mu_f(x/a_m) \to \mu_f(0) = A;$$

this last limit exists in the sense of measures (that is, pointwise almost everywhere) by the assumption that μ_f is approximately continuous at the origin and since the a_m are just real scalars (as in Tukia's argument [17, Theorem G], we just need that the measure of the set where $\mu_f(x/a_m)$ and $\mu_f(0)$ differ by more than δ tends to zero as $m \to \infty$ and this follows immediately from the definition of approximate continuity). As in the parabolic case, the good approximation theorem will now imply that $\mu_g \equiv A$ almost everywhere. Thus g generates a countable uniformly $K(\langle f\rangle)$-quasiconformal group and has matrix dilatation which is constant almost everywhere. This implies that g is affine, and since $g(0) = 0$, g is in fact linear. The proof now concludes exactly as in the parabolic case, since the group $\langle g\rangle$ is an affine $K(\langle f\rangle)$-quasiconformal group.

It is worthwhile contrasting the assumptions of our two theorems above with those of Tukia [17, Theorem G and H2]. In these results, Tukia assumes that the measurable invariant conformal structure is approximately continuous at a radial limit point or continuous at any limit point. Since loxodromic fixed points are radial limit points and since parabolic fixed points are limit points the result seems somewhat similar. However, there are both loxodromic and parabolic C^∞ diffeomorphisms of the sphere which generate discrete quasiconformal groups and for which the invariant conformal structure given by the above construction is not approximately continuous at the fixed points. One would need to establish that the infinite family of matrix dilatations of the powers of f were "approximately equicontinuous" so that the average varied in an approximately continuous fashion to apply Tukia's result.

From Lemma 3.3 and the above theorem we obtain (as for the parabolic case above) the following

3.12 Corollary. *Let f be a loxodromic quasiconformal diffeomorphism of $\overline{\mathbf{R}}^n$. Then f is conjugate to a loxodromic Möbius transformation by a K-quasiconformal homeomorphism for which*

$$K(\langle f\rangle)^{1/2} \leq K \leq \sqrt{(n-1)}K(\langle f\rangle)^{1+1/\sqrt{2}}.$$

3.13 Corollary. *Let f be a loxodromic quasiconformal homeomorphism of $\overline{\mathbf{R}^n}$ which is conformal near a fixed point. Then f is conjugate to a loxodromic Möbius transformation by a K-quasiconformal homeomorphism for which*

$$K(\langle f\rangle)^{1/2} \leq K \leq K(\langle f\rangle).$$

It is a rather nice fact, first observed by Freedman and Skora [3] (with a slightly different formulation) that if G is a group of diffeomorphisms acting on $\mathbf{S}^n$ which is properly discontinuous in the complement of a set E and (each element of G is) conformal in a neighborhood of E, then G is uniformly quasiconformal. In general F will be the limit set and there will be a neighborhood of E in which every generator is conformal. This observation (which is not difficult to establish) enabled them to construct their beautiful example of which we will have more to say later.

We point out that in all of the above examples the conjugating map need not be differentiable. Indeed it is quite easy to construct examples where the conjugacy cannot be differentiable at a fixed point of the parabolic or loxodromic quasiconformal mapping, for instance: let r be a smooth increasing function on $\mathbf{R}$ which is identically equal to two in a neighborhood of 0 and identically equal to three in a neighborhood of ∞. Then $f(x) = r(|x|)x \colon \overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$ is a loxodromic quasiconformal diffeomorphism of $\overline{\mathbf{R}}^2$ which is not conjugate to any Möbius transformation by any homeomorphism which is differentiable in a neighborhood of both 0 and ∞.

In summary, we have shown that the infinite cyclic subgroups of a discrete smooth, uniformly quasiconformal group are quasiconformally conjugate to the appropriate Möbius transformations. We now turn to the general case and the main theorem.

4. Smooth, admissible quasiconformal groups

The concept of admissibility will be purely algebraic. To define it we will need to recall some terminology.

4.1 Definition. Let G be an abstract group. If g and h are elements of G we define the *commutator* of g and h as

$$[g,h] = g^{-1}h^{-1}gh.$$

If H is a subset of G we define the commutator of G and H as

$$[G,H] = \{[g,h]\colon g \in G \text{ and } h \in H\}.$$

It is important to realize that $[G,H]$ is just a subset of G and not a subgroup even if H is a subgroup. The center of G, denoted $Z(G)$, is the largest subset of G such that $[G, Z(G)] = \{\mathrm{Id}\}$. It is not difficult to see that $Z(G)$ is a subgroup of G and that, when G is abelian, $Z(G) = G$. We say that a subgroup H of G is *virtually central* if the set $[G,H]$ is finite.

4.2 Admissibility. An abstract group G is *admissible* if there is an infinite cyclic subgroup which is virtually central.

Henceforth, we will denote an abstract infinite cyclic subgroup by $\mathbf{Z}$. The condition of admissibility implies there is a $\mathbf{Z}$ in G for which $[G,\mathbf{Z}]$ is a finite set.

It will turn out that in fact if there are two or more elements of infinite order in a smooth admissible quasiconformal group, then that group is conjugate into the euclidean group. Thus we would like to see that there are many admissible subgroups of the euclidean group $E(n)$, besides the abelian groups. We note that if H is a subgroup of $E(n)$, then $H\times\mathbf{Z}$ is an admissible subgroup of $E(n+1)$, where the action of $H\times\mathbf{Z}$ is the obvious one given by a translation in the $(n+1)$th coordinate. To obtain a more general result we need some notation.

If f is an element of $E(n)$, then f can be written in the form

$$f(x) = Ax + a, \quad \text{where } A \text{ is orthogonal and } Aa = a. \tag{4.3}$$

Possibly $a = 0$. We say f is proper if A is periodic and $a \neq 0$. The Bieberbach Theorems imply that the elements of a crystallographic group are proper. The following gives a simple geometric characterization of some admissible subgroups of $E(n)$.

4.4 Theorem. *Let G be a discrete subgroup of the euclidean group generated by $g_0, g_1, g_2, \ldots$ where*

$$g_i(x) = A_i x + a_i \quad \textit{and} \quad A_i a_i = a_i$$

(possibly some a_i's are equal to zero). Suppose that g_0 is proper and that for all $i = 1, 2, \cdots$, $A_i(a_0) = a_0$. Then G is admissible.

Proof. Since g_0 is proper, there is an integer m such that A_0^m=Identity, and since $a_0 \neq 0$, g_0 has infinite order. It is now immediate that the infinite cyclic subgroup $\langle g_0^m(x) = x + ma_0 \rangle$ is central.

The hypotheses of Theorem 4.4 are quite restrictive for discrete subgroups of $E(2)$. However they are not so in higher dimensions, where the condition asserts only that each generator preserve one specific direction so the fixed point set of the orthogonal part of any element of G contains the line which a_0 spans. Generically, this will mean the fixed point set contains a 2-plane in which a_0 lies.

We have chosen to call the next result our main theorem. One should note that the hypothesis of admissibility is used crucially in (4.7). Also in view of the more precise results of Theorems 3.6 and 3.11 differentiability is not strictly necessary, for instance one could just assume that every element was affine at fixed points.

4.5 Theorem. *Let G be a discrete, admissible group of quasiconformal diffeomorphisms of $\overline{\mathbf{R}}^n$. Then G is conjugate to a Möbius group by a K-quasiconformal homeomorphism for which*

$$K(G)^{1/2} \leq K \leq \sqrt{(n-1)}K(G)^{1+1/\sqrt{2}}.$$

Proof. Since G is admissible we may find an element f of G such that f is of infinite order and $\langle f \rangle$ is virtually central. We thus define the finite set

$$F = [G, \langle f \rangle].$$

Because G is a discrete quasiconformal group, f is either parabolic or loxodromic and we may assume (by conjugating G with a Möbius transformation) that f fixes infinity and that if f is loxodromic, then infinity is the repulsive fixed point. If f is parabolic we proceed as in the proof of Theorem 3.6 to find conformal maps β_m such that for some subsequence $\beta_{m(j)}$ we have $\beta_{m(j)} \circ f^{m(j)}$ converging uniformly to a $K(G)$-quasiconformal mapping h. While if f is loxodromic we proceed as in the proof of Theorem 3.11 to find such maps. As before $h^{-1} \circ f \circ h$ will be an affine mapping. Let g be an arbitrary element of G. We will show that $h^{-1} \circ g \circ h$ is affine by showing that it has constant matrix dilatation. Again, by the uniform convergence of the sequence $\{\beta_{m(j)} \circ f^{m(j)}\}$ we see

$$g' = h^{-1} \circ g \circ h = \lim \beta_{m(j)}^{-1} \circ f^{-m(j)} \circ g \circ f^{m(j)} \circ \beta_{m(j)}. \tag{4.6}$$

Now as $\langle f \rangle$ is virtually central we see that for all j there is $\eta_j \in F$ such that the mapping

$$f^{-m(j)} \circ g \circ f^{m(j)} = g \circ n_j. \tag{4.7}$$

Since F is a finite collection of mappings, we may pass to another subsequence which we will simply denote by $\{f^{m(k)}\}$, such that

$$f^{-m(k)} \circ g \circ f^{m(k)} = g \circ \eta$$

for some fixed η in the set F. From (4.6) we then find

$$g' = h^{-1} \circ g \circ h = \lim \beta_{m(j)} \circ g \circ \eta \circ \beta_{m(j)}^{-1}$$

is a $K(G)$-quasiconformal homeomorphism of $\overline{\mathbf{R}}^n$, since $g \circ \eta \in G$ and since the β_m's are conformal. Calculating the matrix dilatations of elements of this sequence and using the fact that the matrix dilatation of $g \circ \eta$ is continuous everywhere, and so in particular at infinity and at the origin, then applying the good approximation theorem as we did in Theorems 3.6 and 3.11 we see, as before, that

$$\mu_{g'} = \lim \mu_{g\eta}(\beta_{m(k)}(x)) = A \in S(n)$$

(notice here we only need that $g \circ \eta$ is affine at infinity). Thus g' is a $K(G)$-quasiconformal affine mapping. Since g was arbitrary, the group $h^{-1} \circ G \circ h$ is affine group which is $K(G)$-quasiconformal. It is then conjugate to a conformal group via a linear mapping from the results of §2. The theorem and the estimate on the dilatation now follow as before.

We note that actually the dilatation of h is no more than that of f, thus the possibly better estimate $K \leq \sqrt{(n-1)}K(f)K(G)^{1/\sqrt{2}}$ is valid.

4.8 Corollary. *Let G be a finitely generated infinite abelian, discrete quasiconformal group of diffeomorphisms of $\mathbf{S}^n$. Then G is quasiconformally conjugate to a discrete Möbius group. Consequently the rank of G is at most n.*

This in particular implies that the example constructed by Scott and Tucker of a $\mathbf{Z}$ acting effectively on euclidean 3-space not topologically conjugate to a translation cannot be conjugate to a smooth quasiconformal action (see [4] for other such examples.)

4.9 Corollary. *Let G be an admissible discrete quasiconformal group of diffeomorphisms of $\overline{\mathbf{R}}^n$. Then G is elementary.*

Consequently if G has two or more independent elements of infinite order (that is these elements of infinite order do not generate an infinite cyclic subgroup), then G is quasiconformally conjugate to a subgroup of $E(n)$ and therefore has an abelian subgroup of finite index.

Proof. Rewriting equation (4.7) we find

$$g \circ f^{m(j)} = f^{m(j)} \circ g \circ \eta_j,$$

where η_j lies in a finite set. We can assume that infinity is an attractive fixed point if f is loxodromic (or otherwise consider f^{-1}). This equation easily implies that g fixes infinity and hence, because g is arbitrary, every element of G fixes infinity (see Definitions 3.4 and 3.10). Since G stabilizes a point, G is elementary (see [4]). The final statement follows from the classification of elementary Möbius groups which stabilize infinity (if G has more than two independent elements of infinite order then G is purely parabolic) and the Bieberbach Theorems.

We now consider the hypotheses of Theorem 4.5 and give examples to show that they are essentially all necessary. As we mentioned in the introduction, the assumption that there is an element of infinite order in the group is necessary, when $n \geq 4$, due to the existence of smooth counterexamples to the generalized Smith conjecture. In dimension two this is unnecessary as two dimensional quasiconformal groups are automatically quasiconformally conjugate to conformal groups. When $n = 3$, finite groups of diffeomorphisms are known to be conjugate to conformal groups provided each element has a fixed point (which then implies the fixed point set is an unknotted circle by the affirmative solution to the Smith Conjecture; see [13]).

Freedman and Skora [3] have constructed a uniformly quasiconformal group of diffeomorphisms of $\mathbf{S}^3$ which is isomorphic to a finite extension of a free group of large rank and which is not even topologically conjugate to a Möbius group. Furthermore, they conjecture that the free part of their action is also not topologically conjugate to a Möbius group. This implies that some general algebraic restriction is necessary. There are also examples in [4, I and II] of uniformly quasiconformal groups isomorphic to the free product $\mathbf{Z} * \mathbf{Z}_p$ (here $\mathbf{Z}_p$ is the finite abelian group of order p) acting on $\mathbf{S}^n$, $n \geq 4$, and not topologically conjugate to Möbius groups. These groups are obtained by using the quasiconformal versions of the Klein-Maskit combination theorems of [11] to combine a "bad" elliptic element with a Möbius transformation.

We do not know if the hypothesis of admissibility in Theorem 4.5 can be replaced by the hypothesis that G is virtually abelian (notice that these two hypotheses are independent). It may be that this is not the case. It seems possible that there is a smooth quasiconformal group containing an exotic involution and isomorphic to the Dihedral group; we will pursue this elsewhere. We point out that there is an example in [11] of a uniformly quasiconformal group isomorphic to $\mathbf{Z}^{n-1}$ (and so of course abelian), acting on $\mathbf{R}^n$, which is not quasiconformally conjugate to a Möbius group. As remarked in that paper, the group can be made to be smooth except at one point. It cannot be made smooth at this last point by our results above. Furthermore, as remarked in [12], the dilatation of this group can be assumed arbitrarily close to one.

Finally we observe that if G is a cocompact discrete quasiconformal group isomorphic to a crystallographic group Γ, then G is topologically conjugate (actually quasiconformally if $n \neq 4$) to Γ. In view of Theorem 4.4 and this remark, there is rather wide class of abstract euclidean groups for which a smooth discrete quasiconformal group isomorphic to such a group is actually conjugate into the euclidean group.

It is worth observing the following consequence of Theorem 4.5. Recall we have just observed that bad elliptics can lie in some infinite discrete quasiconformal groups. However, we see

4.10 Proposition. *Let σ be a periodic diffeomorphism of $\mathbf{S}^n$ of period p. If σ is not topologically conjugate to an orthogonal transformation, then there is no discrete smooth $\mathbf{Z} \times \mathbf{Z}_p$ action on $\mathbf{S}^n$ which contains σ and is of bounded distortion.*

Actually, of course, a periodic diffeomorphism which is not conjugate to an orthogonal transformation can lie inside no finitely generated infinite abelian group of bounded distortion acting on $\mathbf{S}^n$. Finally, we restate our main theorem in terms of representations of abstract groups in $\mathrm{Diff}(\mathbf{S}^n)$, the group of diffeomorphisms of $\mathbf{S}^n$.

4.11 Theorem. *Let G be an abstract admissible group and $\Phi\colon G \to \mathrm{Diff}(\mathbf{S}^n)$ a discrete and faithful representation. Then $\Phi(G)$ is topologically conjugate by a homeomorphism of bounded distortion into a subgroup of $\mathrm{Möb}(n)$ if and only if $d(\Phi(G))$ is essentially bounded.*

References

[1] L. V. Ahlfors, *A somewhat new approach to quasiconformal mappings in $\mathbf{R}^n$*, Complex Analysis, Kentucky, Lecture Notes in Math., Vol. 599, Springer, Berlin, 1977.

[2] H. Federer, *Geometric measure theory*, Springer, New York, 1969.

[3] M. Freedman & R. Skora, *Strange actions of groups on spheres*, J. Differential Geometry **25** (1987) 75–98.

[4] F. W. Gehring & G. J. Martin, *Discrete quasiconformal groups*, Proc. London Math. Soc. (3) **55** (1987) 331–358.

[5] C. H. Giffen, *The generalized Smith conjecture*, Amer. J. Math. **88** (1966) 187–198.

[6] F. P. Greenleaf, *Invariant means on topological groups*, Van Nostrand Math. Studies, 1969.

[7] M. Gromov, *Hyperbolic manifolds, groups and actions*, Ann. of Math. Studies, No. 97, Princeton University Press, Princeton, NJ, 1980, 183–213.

[8] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.

[9] H. Maass, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Math., Vol. 216, Springer, Berlin, 1971.

[10] G. A. Margulis, *Free totally discontinuous groups of affine transformations*, Soviet Math. Dokl. **28** (1983) 435–439.

[11] G. J. Martin, *Discrete quasiconformal groups that are not the quasiconformal conjugates of Möbius groups*, Ann. Acad. Sci. Fenn. Ser. AI **11** (1986) 179–202.

[12] J. McKemie, Ph. D. Thesis, University of Texas at Austin, 1986.
[13] J. W. Morgan & H. Bass, *The Smith conjecture*, Academic Press, New York, 1984.
[14] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Ann. of Math. Studies, No. 97, Princeton University Press, Princeton, NJ, 1980, 465–496.
[15] P. Tukia, *On two dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. AI **5** (1980) 73–78.
[16] ——, *Differentiability and rigidity of Möbius groups*, Invent. Math. **82** (1985) 557–578.
[17] ——, *On quasiconformal groups*, J. Analyse Math. **46** (1986) 318–346.
[18] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math., Vol. 229, Springer, Berlin, 1971.
[19] J. A. Wolf, *Spaces of constant curvature*, Publish or Perish, Boston, 1984.

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